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FRACTURE MECHANICS ASSESSMENTS AND DESIGN

Elastic–plastic aspects of fracture stress analysis: methods for other than standardized test conditions

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The J contour integral analysis, despite certain limitations, appears to be the most embracing engineering theory of fracture currently available. It is seen as most relevant for cases where high constraint is maintained as the extent of plasticity increases because it is that case that can be related to the linear elastic fracture mechanics (l.e.f.m.) datum case. Stable crack growth is described in terms of plastic work absorption rate rather than as an increase in toughness of a metallurgical nature so that the R-curve expressed in terms of J is seen as work dissipation rate normalized in terms of shape and size factors b and η. Unstable ductile crack growth can be described in terms of an imbalance of deformation, energy rate or characterizing terms, all giving the same results for small amounts of crack growth, with the material tearing resistance described by $T_{mat} = (E/\sigma_y^2)(dJ/dA)_{mat}$. A J-based design curve is described, analogous to the well known crack opening displacement (c.o.d.) design curve. The use of an effective toughness beyond that found at the onset of initiation without the complexity of a full instability analysis is outlined. Methods for avoiding unstable ductile tearing before a stated degree of plastic collapse are now available, although the circumstances when a change to a different micro-mode of separation might occur are still not describable in continuum mechanics terms.

Introduction

The application of yielding fracture mechanics to problems of structural design will be discussed in terms of the J contour integral. Although the main use of J has been for material toughness testing, so that the fracture toughness can be derived from pieces that yield, it has been argued elsewhere (Burdekin & Harrison 1979; Turner 1979*b*, 1980*a*) that most other theories applied to the fracture of engineering components can be related to J. The implications of expressing data in terms of J are therefore examined first. Methods of applying J to some problems of design are then discussed in relation to both the onset of crack growth and the period of slow growth that follows. Throughout, the meaning of the term G as energy release rate is restricted to the circumstances of linear elastic fracture mechanics (l.e.f.m.). Departure is also made, where appropriate, from the strict mathematical definition of J, itself relevant only for nonlinear elastic (n.l.e.) material.

The path-independent contour integral, J, measuring the strength of the singularity in energy density at a discontinuity in a two-dimensional stress and strain field was developed independently by Eshelby (1956), Cherypanov (1967) and Rice (1968*a*). Recognition of a possible relevance to fracture problems arose in Rice's presentation where the meaning of J as potential energy change per unit crack extension, $-dP/Bda$, appeared as a generalization of G.

The relation to the crack opening displacement (c.o.d.) model of Dugdale (1960) or Bilby *et al.* (1964),

$$J = f\delta, \tag{1a}$$

where δ is c.o.d. and f is the restraining stress in the yielded strip, was also pointed out. The role of J in characterizing the singularities in crack tip stress and strain fields in n.l.e. material was brought out by Hutchinson (1968) and Rice & Rosengren (1968) in the so-called H.R.R. solutions for a power law hardening material. Re-expression of these result by McClintock (1968) gave explicit equations for crack tip stress and strain in terms of J and the hardening exponent n.

The theoretical basis of elastic–plastic fracture mechanics (e.p.f.m.) was reviewed extensively by Rice (1968*b*). Computational studies with the use of the finite element method to examine the path independence of the J integral when using incremental plasticity were conducted by Hayes (1970), who emphasized the characterizing as distinct from the energetic meaning for dissipative materials. Similar studies were also made by Boyle (1972) and Sumpter (1973), who for many cases related the computed values of J and δ by

$$J = m\sigma_y\delta; \quad 1 \leqslant m \leqslant 3. \tag{1b}$$

Fracture tests in the elastic–plastic régime were conducted by Begley & Landes (1972) and Landes & Begley (1972), with the use of an experimental determination of J from the difference in work done, $-\mathrm{d}w$, per unit crack extension, $B\mathrm{d}a$, for cracks cut sequentially to longer initial lengths. Early applications to the onset of fracture were summarized by Knott (1973). More recent developments have been discussed in Latzko (1979). Such work is not reviewed here in detail but rather drawn on selectively as required to support the theme under discussion.

The implications of a J-based model of fracture

(*a*) *Characterization of onset of separation*

Aside from the obvious complexity of the use of plasticity theory, the major difficulty in studying e.p.f.m. is to identify the onset of separation and to measure the amount of slow growth that occurs subsequently. In the absence of a low energy micro-mode of separation, such as cleavage, immediate instability is prevented by the spread of plasticity, which may give rise to the development of shear-lips at the fracture surface, and to more widespread deformation. The former effect occurs even in the l.e.f.m. régime when testing thin sheet, but with thick sections unstable fracture usually follows closely after onset of separation in the l.e.f.m. régime to give a geometry-independent value of fracture toughness, denoted G_{Ic} or K_{Ic} (where $K^2 = \bar{E}G$). With plasticity, despite conditions of plane strain at the mid-thickness, the increase of plastic dissipation with crack length will usually far outweigh the increase of applied severity with crack length so that stable crack growth occurs, driven only by a rising load until some final condition for unstable growth is reached. The most certain way to determine onset of separation is a multiple test-piece technique where successive tests are stopped at various degrees of slow crack growth and the data extrapolated back to the so-called crack-opening stretch line (figure 1), where the material has deformed intensely but not yet separated. J test procedures have recently been summarized by Begley & Landes in Latzko (1979). It should be clear that even if plane strain exists at the central section of the test-piece, the value of J determined at onset of growth, J_i, although conceptually the same term as G_{Ic}, would not in general be equal in value. The reason for this is that the l.e.f.m. procedure allows a certain small departure from linearity. If this is caused by slow growth, albeit small in relation to test-piece width, it will be of an absolute extent larger than the c.o.d., whereas the J procedure attempts to define the onset of separation with no slow growth other than the crack opening stretch region. Nevertheless, the purpose of J testing is usually to derive a value of toughness that could be used with a l.e.f.m. based design procedure such as the A.S.M.E. Boiler and Pressure Vessel Code (1977). One stimulus to the desire to test small pieces was the need to study pre-existing samples of steel already under irradiation in service plant. The size of deep-notch test-piece required for J tests is usually taken as

$$B,\ b > (25 \text{ or } 50)\ J_{\mathrm{Ic}}/\sigma_{\mathrm{y}}, \tag{2a}$$

following suggestions by Paris in discussion of Begley & Landes (1972). This is smaller than the well known l.e.f.m. requirement for plane strain,

$$B,\ b,\ a \geqslant 2.5(K_{\mathrm{Ic}}/\sigma_{\mathrm{y}})^2, \tag{2b}$$

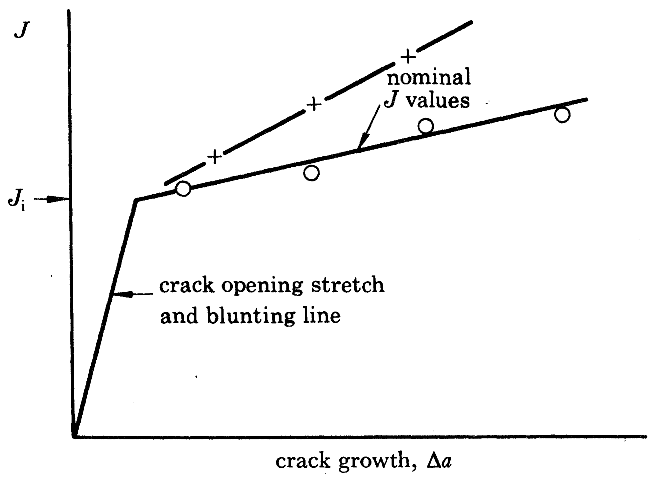

FIGURE 1. Conventional multi-test-piece procedure for determining the value of J at onset of crack growth by extrapolation to the crack blunting line: schematic of two configurations extrapolating to the same value of J_i.

by approximately σ_{y}/E. Infiltration of the crack by a hardening resin and subsequent sectioning (Robinson & Tetelman 1974) demonstrated that such dimensions were indeed adequate in deep-notch three-point bend pieces (d.n.b.) for the mid-section deformation to be representative of plane strain, although lingering doubt may remain on whether a full plane strain constraint of stress is attained. A further requirement to ensure plane strain when yield is extensive appears to be

$$b \leqslant B \tag{3}$$

so that, particularly in low hardening materials, a plane strain slip field is set up. The preferred test-pieces for J testing are deeply notched three-point bend or compact tension (c.t.). For several geometries simple formulae of the form

$$J = \eta w/Bb \tag{4a}$$

have been developed by Rice *et al.* (1973), where the value of work done, w, is taken before the onset of separation. The term $\eta = 2$ is given by Rice *et al.* for the d.n.b. piece in pure bending.

Merkle & Corten (1974) analysed the compact tension (c.t.) piece. However, in the l.e.f.m. régime the relation

$$G = \eta_{el} w_{el}/Bb \tag{4b}$$

also exists (Turner 1973). For d.n.b. with span $S = 4W$, $\eta_{el} \approx 2$ so that an overall relation can be used for conventional three-point bend test-pieces.

$$J \approx \eta_0 w_0/Bb, \tag{4c}$$

with $\eta_0 = 2$ (Srawley 1976). The near identity or otherwise between the elastic, plastic and overall values of η will be referred to below.

The c.o.d. procedure followed a rather different philosophy. Early usage (e.g. U.K.A.E.A. 1969) was for general structural engineering steels at a temperature for which transition to cleavage behaviour might occur. A strong preference therefore emerged for the use of full thickness test-pieces and for a high constraint configuration such as d.n.b. The d.n.b. piece on which the method standardized (B.S.I. 1979) was also well suited for simple analysis before elastic–plastic computational methods were available. Some choice is left to the user, however, on whether critical c.o.d. is measured at onset of separation or after some slow growth or even at maximum load, on the general grounds that restriction to onset of separation is too conservative for many applications. Slow growth will be discussed later in the paper.

The difference in viewpoint between the proponents of the two schools over size of test-piece is not fundamental to the concept adopted but to the usage envisaged and also to the question whether in b.c.c. metals the temperature at which transition to cleavage occurs is itself size-dependent if measured in plane strain. If it is, then the full-thickness c.o.d. procedure is more representative of severe service conditions than the small test-pieces usually used in J testing, although both advocate use of configurations giving high constraint. The proponents of testing small pieces argue that the apparent effect of size reflects scatter of the weakest link type (Landes 1979) so that full thickness data correspond closely to the lower bound of the small test-piece data. In that view there is no real effect of size on the temperature for the appearance (i.e. micro-mode) transition in plane strain, although it is agreed that the ductility transition temperature for fracture before or after extensive yield (without change of micro-mode) is size dependent, and of course, that size affects the degree of plane strain or plane stress induced in a test-piece. A full thickness test procedure expressed in terms of J has been advocated (Sumpter & Turner 1976).

It remains to be seen whether, or under what circumstances, either c.o.d. or J or some other single term is an adequate measure of onset of fracture. It was suggested (Turner 1980*a*) that any one-parameter description of separation, if interpreted rigorously, implies that either the stress and strain state around any crack is unique in pattern and varies only in magnitude or that any feature of which the parameter is independent does not affect separation. The lack of uniqueness between plane strain and plane stress is obvious, but within either the stress field is unique for l.e.f.m. and varies only in magnitude. The ratio $\sigma_x : \sigma_y : \sigma_z$ just ahead of a crack is $1:1:2\nu$ for plane strain. The H.R.R. solutions similarly give a field that is unique in plane strain for a given value of the hardening exponent n varying from $1:1:1$ for $n = 1$ to $\pi : \pi+2 : \pi+1$ (approx. $0.6:1:0.8$) for $n = 0$. This latter is identical to the Prandtl slip field for contained yield and comparable to the net section yield slip fields that contain slip lines which curve through $\frac{1}{2}\pi$ degrees (e.g. deep-notch d.n.b. and d.e.n.), for a rigid-plastic material. Such high triaxality is also comparable to any problem with very high hardening (tending to $n = 1$ in the limit) but not to cases of low hardening where the slip pattern is straight or only slightly curved. Support for this argument is found in computational studies where the one-to-one relation between J values and crack tip deformation remained for d.n.b. even when plasticity became extensive but was gradually lost for c.c.t. (Sumpter & Turner 1976*a*). The same trends were found even more clearly in terms of the stress fields by McMeeking & Parks (1979), who used a large geometry change isoparametric element that gave a much improved representation of the crack tip details.

At least two parameters would be required to describe fields that differed in both intensity and triaxality. Thus, unless the criterion of separation is insensitive to triaxality, it seems implausible that a one-parameter measure of intensity alone should be an adequate description of fracture, as was indeed pointed out by McClintock (1965). In so far as a one-parameter description is used, be it J or c.o.d. or other, it must relate the extensive yield case to a datum case. The universally agreed datum is plane strain l.e.f.m. That is a case with high constraint ($1:1:2\nu$) so that it is the high-constraint plasticity cases that are most likely to follow a one-parameter description of onset of fracture. The point here is not that J is invalid for conditions of low constraint but that there is no suitable datum within l.e.f.m. to which the low constraint cases can be related. The extent to which J or any similar parameter provides a measure of some effective intensity relevant to fracture in stress fields that are in fact different will depend on the micro-mechanism of separation, and experiment must be the guide. Although there have been many studies of the effects of geometry on the onset of fracture, such as Egan (1973), Griffis (1975), Logsdon (1976), Robinson (1976) and many others, much uncertainty remains.

(*b*) *Slow stable growth*

When R-curves are used to describe stable crack growth in l.e.f.m. there is no doubt that the increase in apparent toughness is caused by the development of shear lip which reflects an extent of plasticity large in relation to the thickness while small in relation to the ligament width. This is not necessarily so in plasticity. A general picture of stable growth in ductile micro-mode was presented by Green & Knott (1975). This showed crack blunting before initiation at a representative value of c.o.d., δ_i, followed by continued growth of c.o.d. at the original tip from which extended a region with more or less constant flank angle leading to the actual advancing tip. By making infiltration castings of the slowly advancing tip, Garwood (1976) confirmed this picture, also finding for a C–Mn steel that the 'final' advancing tip appeared as a small value of c.o.d., δ_a (figure 2*a*). This tip opening, δ_a, remained constant with crack growth over the extension observed (figure 2*b*). For that particular steel $\delta_i/\delta_a \approx 4$ for Δa up to 4 mm in some tests. These tests and later confirmation by Willoughby (1979), using a grain growth technique, showed no evidence on the microscale of any increase of toughness with crack growth. However, expressed in terms of J (from (4)), toughness did indeed increase to give a rising R-curve, such as figure 3. Even with the shear lip eliminated by side grooving to give flat fracture (still ductile in micro-mode), the R-curve showed an increase in the apparent J value of some sixfold for the particular steel (figure 3, curve 1). It is clear that shear lip is not the sole or even dominant cause of the rising R-curve, which is in fact a measure of continued dissipation of plastic work in the whole component. Thus, the nature of the 'toughness' that is increasing is plastic dissipation, of which the true surface energy component is negligible and cannot be separately identified (Rice 1965). It is not therefore clear what meaning should be attached to J once the use of plasticity materials and acceptance of slow growth requires distinction between the various meanings of characterization, stress field

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energy absorption rate and energy release rate that are all relevant for n.l.e. material. The arguments are not clear cut. A characterizing meaning may prove to be necessary in the sense that the R-curve will not be geometry-independent unless crack growth starts at the same condition in a unique original crack tip field, and grows in the same way in a field common for any advancing crack. Hutchinson & Paris (1977) have argued that a J-dominated crack tip stress field remains for small amounts of crack growth in plasticity such that

$$\omega \equiv (b/J)(\mathrm{d}J/\mathrm{d}a) \gg 1. \tag{5}$$

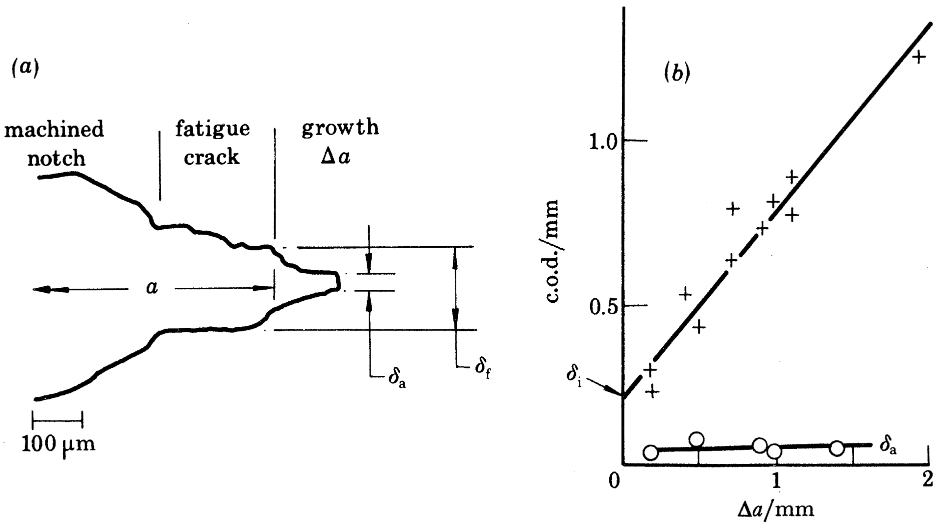

FIGURE 2. Crack opening displacement determined by infiltration technique after slow stable crack growth: (*a*) mid-layer section of infiltration casting; (*b*) c.o.d. at the tip of the original fatigue crack, δ_f, and at the advancing crack, δ_a. Data for a C–Mn steel in bending with fracture in ductile micro-mode.

Rice (1978) has derived a logarithmic form for the singularity around an advancing crack on arguments of possible equilibrium stress fields and writes the opening of the advancing crack, Δ, as

$$\Delta = \frac{\alpha r}{\sigma_y}\frac{\mathrm{d}J}{\mathrm{d}a} + \frac{\beta r \sigma_y}{E}[1+\ln (R/r)], \tag{6}$$

where r is the polar distance from the tip, α and R are undetermined parameters and β is a constant (*ca.* 3.93 for $v = 0.3$). Rice suggests that R is of the order of the maximum extent of plastic zone, and α might be comparable with $1/m$ (where m is defined in (1b)). As discussed by Turner (1979*c*), the second term in (6) is negligible except for exceedingly small values of r; for Garwood's data, on which figure 3 is based, the first term of (6) dominates and gives $\alpha \approx 1/1.9$, in good agreement with $m = 2.1$ from his crack initiation data for the same steel and same test configuration. However, computational studies on the advancing crack by McMeeking & Parks (1979) showed that during crack growth the J contour integral remained substantially path independent only in the far field ($J_{\mathrm{f.f.}}$) while the near tip values became path-dependent. From similar studies, Shih *et al.* (1979) concluded that experimental values of J derived (for c.t. pieces) from (4) related to $J_{\mathrm{f.f.}}$. The picture of $\mathrm{d}J/\mathrm{d}a$ so derived by Shih starts at a high and nearly constant value, reduces and then remains nearly constant at a low value. Such a pattern clearly agrees with the R-curve picture of figure 3 where there is a nearly constant initial slope of high value and a final (non-zero) slope of a rather low value.

In analysing the R-curve data such as figure 3, Garwood *et al.* (1978) used a modified form of (4*a*) to account for the difference in meaning of J before and after initiation in the n.l.e. material. The essential relation (figure 4) is

$$dw = dU - BJ da, \tag{7}$$

where dU relates to the increment of work BDEC but dw relates to the change in area under the load deformation curve, which for n.l.e. material will entail the recovery of area OBDO, in

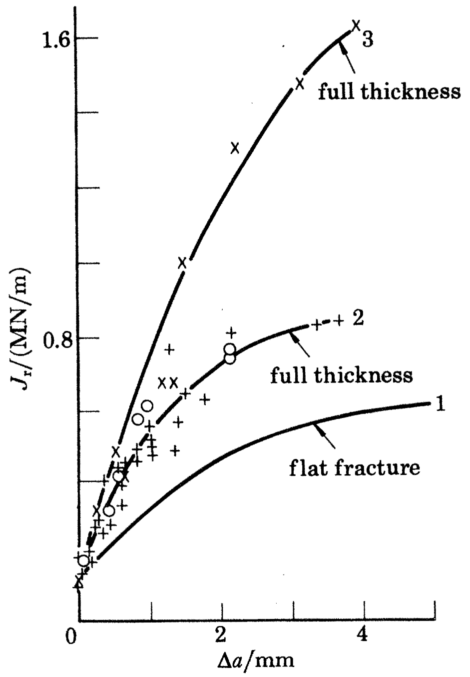

FIGURE 3. Crack growth resistance curves (R-curves) for stable crack growth with extensive plasticity in three-point bend tests: (1) flat fracture, shear lips inhibited by deep side grooves; (2) full thickness tests including shear lips for two different absolute sizes of ligament; (3) full thickness tests with a ligament of proportion different from case 2.

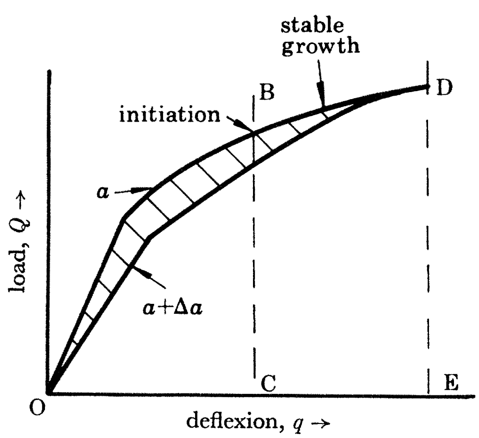

FIGURE 4. The distinction between dU and dw in the definition of J_r. dU = BDEC; $BJ da$ = OBDO; $dw = dU - BJ da$.

itself equal to $BJ\mathrm{d}a$. For linear material the recovery would be $BG\mathrm{d}a$. The conventional determination of the apparent J, J_{app}, after some crack growth, Δa, follows (4 a) and writes

$$J_{\mathrm{app}} = \eta(w + \mathrm{d}U)/Bb. \tag{8}$$

The corrected value based on $(w + \mathrm{d}w)$ was shown (Turner 1980 b) to be

$$J_{\mathrm{corr}} = J_{\mathrm{app}}[1 - \Delta a(\eta - 1)/b], \tag{9a}$$

whence for the d.n.b. piece, where $\eta = 2$ (and approximately so for c.t.),

$$J_{\mathrm{corr}} \approx J_{\mathrm{app}}[1 - (\Delta a/b)]. \tag{9b}$$

Garwood *et al.* (1978) accepted that with growth the J field was soon lost and did not restrict their arguments by the criterion (5) that $\omega \gg 1$. Instead they defined the meaning of $\mathrm{d}J$ subsequent to initiation through the dissipation rate, $\mathrm{d}w/B\mathrm{d}a$, using (7). This term is called J_{r}. One method of deriving J_{r} is by differentiation of (4) (where $\mathrm{d}w$ is assigned the meaning of (7)), and in Turner (1980 a) the work rate was expressed as

$$\mathrm{d}w/B\mathrm{d}a = (J_{\mathrm{r}}/\eta)[(b/J_{\mathrm{r}})(\mathrm{d}J_{\mathrm{r}}/\mathrm{d}a) - f_1(\eta)], \tag{10}$$

where

$$f_1(\eta) = 1 + (b/\eta)(\mathrm{d}\eta/\mathrm{d}a). \tag{11}$$

If

$$\omega \gg f_1(\eta), \tag{12}$$

$$\mathrm{d}w/B\mathrm{d}a \approx (b/\eta)(\mathrm{d}J_{\mathrm{r}}/\mathrm{d}a). \tag{13}$$

The R-curve expressed in terms of J_{r} is defined as

$$J_{\mathrm{r}} = J_{\mathrm{i}} + (\mathrm{d}J_{\mathrm{r}}/\mathrm{d}a)\Delta a, \tag{14}$$

where J_{i} is the value at initiation. From (10),

$$\mathrm{d}J_{\mathrm{r}}/\mathrm{d}a = (\eta/b)(\mathrm{d}w/B\mathrm{d}a) + (J_{\mathrm{r}}/b)f_1(\eta). \tag{15}$$

If the term $(J_{\mathrm{r}}/\eta)f_1(\eta)$ is included, then $J_{\mathrm{r}} \equiv J_{\mathrm{corr}}$ (equation (9)). If the term is neglected because $\omega \gg f_1(\eta)$ then $J_{\mathrm{r}} \equiv J_{\mathrm{app}}$ (equation (8)).

To allow discussion of continuum models of behaviour it is convenient to visualize a material more realistic for plasticity than the n.l.e. material on which J theory is strictly based, while retaining certain ideal characteristics. This notional material, called elastic–plastic–elastic (e.p.e.), follows total theory plasticity on loading but linear elasticity on unloading (Turner 1980 a). It may further be restricted to a so-called ideal case (ideal e.p.e.) in which only proportional loading occurs and the boundary conditions of loading are constant. The intention is to establish for plasticity circumstances in which no doubt arises on the relevance of J during loading, for which in n.l.e. or ideal e.p.e. a constant ratio of crack tip stresses is maintained, followed by unloading, for which ideal e.p.e. is closely representative of real behaviour. Other restrictions, albeit unwanted, may also be necessary for a one-parameter model of fracture. Satisfaction of (12) would imply that in (7) $\mathrm{d}w \approx \mathrm{d}U$ with $BJ\mathrm{d}a$ negligible. When that is so, the distinction between n.l.e. and ideal e.p.e. is lost and J theory should be relevant, a condition previously denoted by (5). For many test-piece cases, (5) (i.e. $\omega \gg 1$) and (12) imply a similar restriction, for example in d.n.b. where $\mathrm{d}\eta/\mathrm{d}a \approx 0$ and $f_1(\eta) \approx 1$. In so far as they differ, for example in shallow notch cases where $f_1(\eta) \approx b/a$, it is suggested that (12) is the more revealing because it accounts for configuration. It contained yield where $J \approx G$, restriction to $\omega \gg 1$ is not necessary because (7) is an adequate representation of both n.l.e. and ideal e.p.e. material. A further modification to the definition of J_r for ideal e.p.e. material is required, if, as seen later, (7) is replaced by (20).

In principle the J_r analysis allows the work dissipation rate in one configuration to be normalized to a reference curve and reapplied to another. The normalization adopted involves both size and shape factors so that the term (b/η) is conceptually similar to the energy 'filter factor' of Broberg (1974). In practice it is still uncertain whether, because of the different degrees of constraint in the different configurations, particularly with low hardening, the R-curve is sufficiently geometry independent to allow the transfer of data between different configurations. The appropriate values of η are also uncertain for some cases, although η can be measured experimentally (Ernst *et al.* 1979). The uncertain effects of configuration and triaxality reflect Rice's comment that some of the terms in his model (equation (6)) are not uniquely determined so that the R-curve may not be geometry-independent. It remains to be seen whether the normalization described is indeed adequate, particularly for cases where the uniqueness of the initiation event is open to doubt. It seems necessary to treat the effect of shear lip separately, since as seen (figure 3), the proportions of the ligament affect the size of shear lip and hence the R-curve. It has also been shown that anisotropy greatly affects the R-curve when propagation occurs in different directions through the plate (Willoughby *et al.* 1978). Thus, for the time being, use of J–R curves is proceeding on the basis of finding either a lower bound curve (i.e. high constraint, no shear lip, crack growth in the least tough direction; see Garwood 1980) or using a thickness, configuration and stress system known to be relevant to the required service.

(*c*) *Unstable crack growth*

To make use of an R-curve analysis, the condition for the final instability must be predictable. Three relatable arguments expressed through J have emerged together with an alternative suggestion. Paris *et al.* (N.R.C. 1977) considered the changes in length of the plastic region at the crack and then of the elastic regions remote from it, as a crack advances. The change in the plastic region was identified with change of c.o.d. and was expressed only at limit load, using, for c.c.t., $\sigma = \sigma_y(W-2a)/W$ and $J = \sigma_y\delta$. The plastic change is thus $\mathrm{d}\delta/\mathrm{d}a = (1/\sigma_y)(\mathrm{d}J/\mathrm{d}a)$ and the change in overall elastic length of a bar of length D, as load is reduced, was written as $\mathrm{d}(\sigma D/E)/\mathrm{d}a$, where the gross stress σ is restricted by the limit state of the net section. Unstable behaviour was taken to occur when the latter exceeded the former, i.e. when

$$D/W > (E/\sigma_y^2)(\mathrm{d}J/\mathrm{d}(2a)). \tag{16}$$

The right side was denoted 'T_{mat}', and if the initial slope of the R-curve was used (as in figure 1 from conventional J testing) the term was called the 'tearing modulus'. The left side was denoted 'T_{app}'. Instability in deep-notch bending and for a buried flaw were considered by the same mechanics of elastic contraction rate exceeding plastic extension rate inferred from c.o.d. The use of the J–R curve arises from expressing $\mathrm{d}\delta/\mathrm{d}a$ as $(\mathrm{d}J/\mathrm{d}a)/\sigma_y$. In effect this neglects m and its derivative in differentiating (1 *b*), although constraint is discussed for a buried crack. Crack growth is also limited to little beyond the initial slope of the J–R curve by the restriction $\omega \gg 1$ (equation(5)).

An energy rate balance for instability was suggested by Turner (1979*a*), following the R-curve study of Garwood *et al.* (1978). The elastic energy release rate for e.p.e. material, termed I, was evaluated at fixed overall displacement. The elastic displacement, q_{el}, can include the compliance of a structure treated as an effective length of component. The change in elastic energy with crash length is the area CBDN (figure 5). The expression for I is

$$BI = BG - \frac{q_{el}}{\eta_0}(\eta_0 - \eta_{el})\frac{\partial Q}{\partial a} = BG\left(\frac{2\eta_0}{\eta_{el}} - 1\right). \tag{17}$$

Bounds can readily be set on I since for a linear elastic system $I = G$ (corresponding to area CBD′ (figure 5)) and for a nonlinear system $I = J$ (area OABD′O) so that $G < I < J$. Various ways of estimating η and hence I are summarized in Latzko (1979) and Turner (1979*a*). If, as slow growth progresses, this energy rate available exceeds the total work dissipation rate, in the absence of other forms of dissipation, unstable growth must occur. The statement of instability is thus

$$I > \mathrm{d}w/B\mathrm{d}a, \tag{18a}$$

where $\mathrm{d}w/B\mathrm{d}a$ is defined by (10) to include both remote plasticity and local fracture work. If the restriction $\omega \gg 1$ is made, this reduces to

$$I > (b/\eta)(\mathrm{d}J_r/\mathrm{d}a). \tag{18b}$$

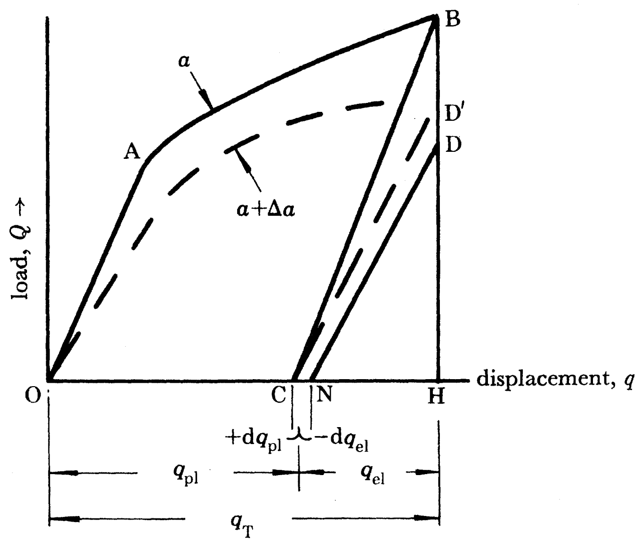

FIGURE 5. The recoverable energy, $BI\mathrm{d}a$, for linear elastic unloading in the presence of extensive plasticity: $BG\mathrm{d}a$ = CBD′; $BI\mathrm{d}a$ = CBDN. For ideal e.p.e. material, $BJ\mathrm{d}a$ = OABD′O.

It is not a true second derivative treatment of instability as proposed by Orowan (1956), but with appropriate restrictions such as a convex R curve, no other dissipation and the inclusion of structural compliance in estimating I, it offers a balance of energy rates for unstable ductile tearing analogous to the Griffith elastic model but using an effective plastic toughness J_{eff} defined by (10) for a component of given b and η values. This effective toughness is material dependent through the slope of the R-curve and component dependent through the factor b/η. The effective toughness of course decreases with crack growth, as shown for a particular structure in figure 6, together with the conventional rising R-curve. Decreasing effective toughness was in vogue as a picture of instability before the rising curve of Krafft *et al.* (1961). From (18*a*), instability occurs when $I > J_{\mathrm{eff}}$. The whole discussion can obviously be expressed in the T notation of Paris by writing (18*b*) in the form

$$(\eta/b)(IE/\sigma_y^2) > (E/\sigma_y^2)(\mathrm{d}J_r/\mathrm{d}a). \tag{19}$$

This is formally identical to the result of Paris but may differ in numerical value according to the particular approximations made in estimating η and hence I (equation (17)), according to the degree of plasticity and the use of n.l.e. or e.p.e. descriptions of material. There is no inherent restriction to a J-dominated stress field in the energy rate balance but the uncertainty on whether the R-curve, however defined, is unique for all configurations and states of stress, remains.

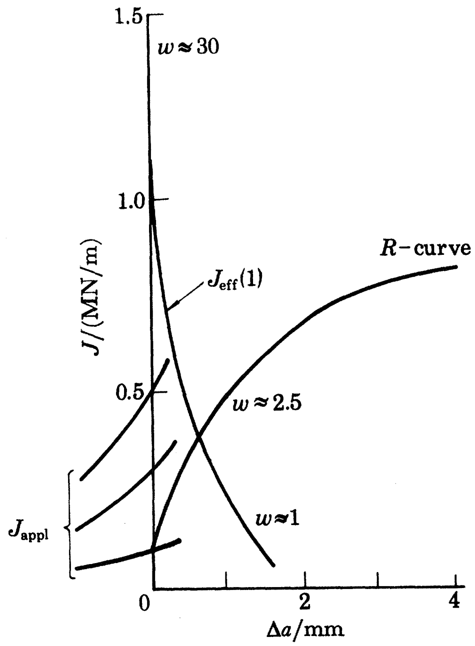

FIGURE 6. The apparent toughness, J_r, rises with the R-curve but the effective toughness J_{eff} (equation (25)) decreases with $\mathrm{d}J/\mathrm{d}a$ as the crack grows. J_{eff} (1) makes use of (13) rather than (10). Also shown are estimates of the applied severity, J_{app}, for the tangency model of instability.

A third presentation used by Garwood (1976, 1979), and later by Shih (1979), adopted the well known tangency between applied severity and the material resistance, analogous to the l.e.f.m. usage of G–R curves but now with both terms expressed in terms of J (figure 6). If a characterizing meaning is given to J for both R-curve and applied severity, application is restricted to $\omega \gg 1$ to ensure that a J-dominated field exists. If an energetic meaning is given to the tangency construction it is the equivalent of taking the energy rate available as J and thus forming an upper bound to I. For purposes of prediction, use of an upper bound on energy available and a lower bound on dissipation will give a conservative estimate of instability.

Some estimates of the actual instability event have been compared with experimental results in which the machine compliance was varied (N.R.C. 1977). As seen (figure 7), the prediction is very satisfactory, but it should be noted that the data are for deep-notch three-point bend tests near the limit load régime with $\omega \gg 1$, and in that régime there is little difference between the various methods of estimating either T_{app} or T_{mat}. It is not clear that such satisfactory estimates can be made for a wider range of circumstances, as seen later.

A rather different approach is described in this Symposium by Harrison & Milne, in which the two-criteria method is extended to predict maximum load as a lower bound on unstable growth. Although the various theories described are based on rather different arguments, they are not as diverse as at first appear. All can be arranged to describe the material property that resists tearing in terms of T_{mat}, i.e. $(E/\sigma_y^2)(dJ/da)$, although there are differences in the most appropriate definition of J and its physical meaning and usefulness when $\omega \gg 1$. This property, even if not completely independent of geometry, represents a different measure of 'tearing toughness' than does the toughness expressed as J_{Ic} or G_{Ic} or δ_{Ic}. Correspondingly, the applied severity can be expressed in terms of T_{app}. The precise terms involved differ according to whether the estimate is made in well contained yield or gross yield and whether an n.l.e., ideal e.p.e. or limit load plasticity model is used. In energetic terms this corresponds to taking the energy release rate as somewhere between G and J, the precise value depending very much on the assumptions made. Thus, even for l.e.f.m. usage, if contained yield is regarded as an ideal e.p.e. case, G should not be corrected for plasticity effects if interpreted as a driving energy rate, although it should be if regarded as an estimate of J in a role characterizing the severity.

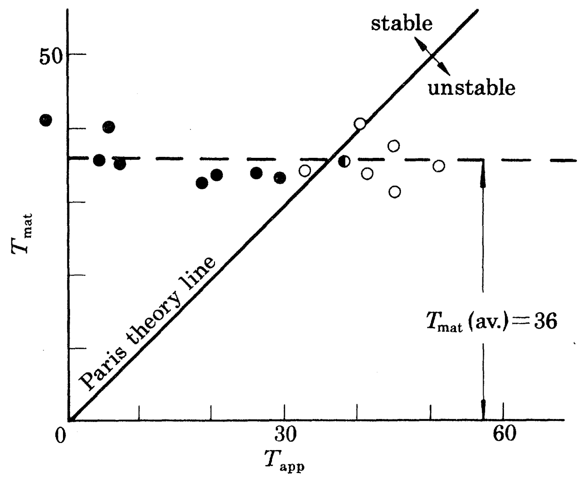

FIGURE 7. Prediction and measurement of unstable ductile crack growth by Paris *et al.* (N.R.C. 1973). Near-limit load behaviour of deep notch three-point bend tests with $\omega \gg 1$, where T_{app} has been altered by placing a leaf spring in series with the test-piece.

It is also clear that, for ideal e.p.e. material, (7) should be rewritten as

$$dw = dU - BI da \tag{20}$$

and the definition of J_r modified accordingly.

A very important restriction to all the foregoing theories is that none accommodate time-dependent effects (logarithmic creep) nor encompass a change of micro-mode (specifically from ductile micro-void coalescence to cleavage) between the determination of the R-curve data, perhaps on a small component, and the application to a real structure.

(c) *Application to design*

The main use of e.p.f.m. for static loading has been to guard against onset of crack growth by ensuring $J < J_i$ or $\delta < \delta_c$. The terminology is rather confused in that, as already seen, a formal definition of J_{Ic} has not yet emerged, and use of J concentrates on 'no' slow growth, i.e. measurement of J_i, whereas use of δ_c may permit some slow growth. For the present, either term, however determined, is taken as a measure of avoidance of initiation of fracture. Explicit acceptance of some degree of slow growth is examined later.

One of the difficulties of applying these concepts to design is the generality of the problem. Indeed, perhaps procedures can only be devised for specific problem areas, thereby accounting for many of the different proposals already existing. Assuming that J is indeed relevant to fracture, either by restricting interest to the configurations that retain high constraint or by accepting J procedure in plane strain as a lower bound estimate of toughness, application requires the two related steps of assessing J in terms of any applied loading of the component and assessing the worst value of the loading to be considered. The former can perhaps be reduced to a simple or 'design curve' relation. The latter must remain problem-related. If (4a) is rewritten

$$J/\eta = w/Bb, \tag{21}$$

it provides a relation between J, geometry and work per unit area, which, in so far as η is independent of degree of deformation (the ideal e.p.e. case), is a unique line with a slope of unity for all cases. Use of (21), with η taken as η_{el} for n.l.e. or ideal e.p.e. behaviour, would then allow some estimate of the load or deformation to cause a certain value of J.

However, a more immediate point is that if a simple expression is sought for J in terms of either load or strain, none can be found except through explicit use of a load–deformation relation. Much of the object of a simple design method is, however, to estimate J when only one or other of the terms of load or strain is known, so that gross simplification must be made and any resulting approximations accepted as the price of simplicity.

The two best known design procedures for avoiding fracture outside the l.e.f.m. range are the engineering c.o.d. method and the two-criteria method. The former expresses severity in terms of strain in the uncracked body and the latter in terms of load, normalized against collapse load. It is perhaps too simple to say the former applies to components that are deformation controlled and the latter to those that are load controlled. Both, of course, are meant to be conservative, yet clearly only guard against certain features implicitly. Thus, assessing limit load, which is a *global* term, even if interpreted for some features such as the nozzle region of a vessel, cannot guard against high *local* strain that might cause fracture, unless the limit state is avoided by an appreciable margin. Conversely, defining a certain post-yield strain level as not inducing an excessive c.o.d. will not automatically ensure that limit load is not attained before the critical c.o.d. is reached. In short, fracture mechanics must be related to other aspects of design, most of which exist as codes for given classes of structures. These also often define the reasonable limits of severity against which protection is sought. General fracture analyses that are supplementary to a code can be outlined but their integration into actual design procedures is a much more detailed issue that can only be attempted for each class of structure separately.

If yield is contained, as surely is intended for most normal design conditions, then estimation of J from G_p (i.e. by using l.e.f.m. and the plastic zone correction factor) appears adequate for many problems, including stress concentration cases (Sumpter & Turner 1976c) if the l.e.f.m. model is chosen carefully. The upper limit of validity for the l.e.f.m. procedure with plastic zone correction as yield becomes less well contained is not clearly defined but extends well beyond an intuitive concept of first yield that for a notched configuration must itself be ill-defined. The evidence is that 'well contained' means well below the net section limit state. Thus, estimates of G_p are valid up to about 0.8 Q_1, where Q_1 is the limit load of the net section, whereafter J increases indefinitely as deformation becomes unrestricted. Replacing the estimate of G_p from l.e.f.m. by the Dugdale ln secant formula, and explicitly bounding the load (which is seen as the controlling variable) by the limit state, closely relates what is here taken as one rule of thumb approximation for J, itself viewed as a single criterion that runs from G at l.e.f.m. to infinity for unlimited deformation, to the two-criteria method.

The second approach is a J version of the c.o.d. design curve. This originated in an estimate of J by Begley *et al.* (1974), with strain as the controlling variable. Separate computational studies of various design type configurations were made by Sumpter (1973). These latter, including cracks at a hole in a plate in tension, cracks in two-dimensional model of a thick cylinder under pressure, and shallow cracks ($a/W < 0.1$) in tension, have been analysed and clarified by supplementary cases (Sumpter & Turner 1976*c*; Turner 1978) as outlined in figure 8, to emerge as a set of equations describing the upper bound of the data (curve J(M),

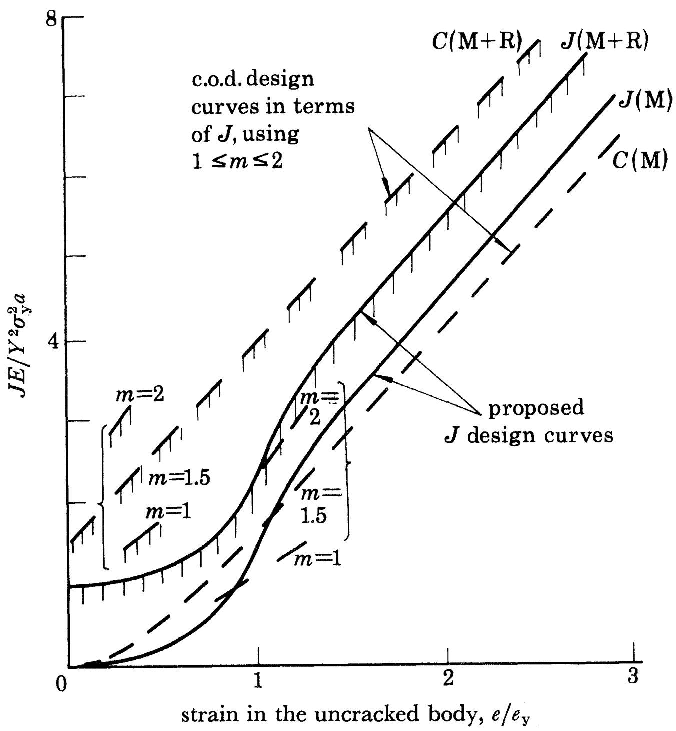

FIGURE 8. The proposed J design curve J(M) and the proposed allowance for yield level local residual stress J(M+R). Also shown is the c.o.d. design curve C(M) or C(M+R) translated to J, using $Y^2a = \pi\bar{a}$, where $\bar{a}$ is the equivalent crack, and $m = 1.5$, in (1 *b*). M = mechanical stress only; M+R = mechanical plus yield level residual stress.

corresponding to the c.o.d. design curve (Burdekin & Dawes 1971; Dawes 1974). The equations were normalized to bring all l.e.f.m. cases to one line by writing the ordinates as J/G_y where $G_y = Y^2\sigma_y^2 a/E$, i.e. the value of G with $\sigma = \sigma_y$. The suggested equations, in terms of the strain in the uncracked body, are:

for l.e.f.m. (neglecting plastic zone correction, for simplicity), $e/e_y < 0.85$,

$$J/G_y = (e/e_y)^2, \tag{22a}$$

for contained yield, $0.85 < e/e_y < 1.2$,

$$J/G_y < 5[(e/e_y) - 0.7], \tag{22b}$$

for uncontained yield, $e/e_y > 1.2$,

$$J/G_y < 2.5[(e/e_y) - 0.2]. \tag{22c}$$

The c.o.d. design curve as expressed by Dawes (1974) was

$$e/e_y < 0.5 \qquad \delta/2\pi e_y a = (e/e_y)^2; \tag{23a}$$

$$e/e_y > 0.5 \qquad \delta/2\pi e_y a = (e/e_y) - 0.25. \tag{23b}$$

This is now interpreted with a meaning of equivalent crack $\bar{a}$ (B.S.I. 1976). Here the two concepts are related (figure 8) by writing $\pi\bar{a} = Y^2 a$ and selecting a value of m in (1*b*) to give curve C(M). Quite close agreement would exist if the value of m were increased from 1 to 2 with increase of strain. The original proposal by Begley *et al.* (1974) falls between the present curve J(M) and the c.o.d. curve translated with $m = 1$, if their constant $1.12\sqrt{\pi}$ is here generalized to Y. It can be questioned whether the J or c.o.d. curve, each in its way based mainly on data for gross yield, is a reasonable picture of the stress concentration case if that is restricted to contained yield. In the absence of more detailed study, it appears that the use of gross yield data compensates approximately for the over simple estimate made for the strain at the stress concentration in the uncracked body. Whether this is fortuitous for the few cases studied or is a reasonable generality is not clear. The equations (22*a–c*) that define the design curve relate to computations in plane strain. This would be over-severe for problems where the load would be restricted by the limit state in plane stress, although the cracked region would experience local plane strain. Basing the estimate of strain on the plane stress load, while retaining the cracked body estimate of J in plane strain, gives a result rather below curve C(M) (figure 8). Studies by Riccardella & Swedlow (1974) and Harrison (1977) demonstrated that many fracture problems would be expected to fall between plane stress and plane strain.

Strictly, J is not relevant to problems of residual stress, but an approximate argument has been made in terms of the strain energy contained in the residual stress field (Turner 1980*b*). Three steps are: (i) identification of the cause of the residual stress to make allowance if need for any effect on toughness; (ii) identification of any reaction stress for treatment as an applied load; (iii) acceptance of the local residual stress as yield level over a volume, V, comparable in extent to the plate thickness extending all along the weld. This third step is then treated by considering a crack parallel to the weld or transverse to the weld, estimating the available strain energy, w, and the relevant η factor for tension of a plate of extent equal to the residual stress region and using these values of η and w to estimate G from (4*b*). This value is then added to the mechanical stress system as an approximate usage of

$$J = J_{el} + J_{pl}, \tag{24}$$

where here $J_{el} \equiv G$ from residual stresses and $J_{pl} \approx J$ (overall) from figure 8 (curve J(M)). The result is to adjust the ordinate by unity on the axis scale of figure 8 (curve $J(M+R)$) or add unity to the right-hand side of (22). This is equivalent to adding a yield stress level case to the object under consideration. The allowance $J/G_y = 1$ is roughly one-third as severe as the c.o.d. procedure of adjusting the abscissa by unity for yield level residual stress. The worst possible combination of stresses in plane strain could provide strain energy of about $\frac{3}{2}\sigma_y^2/E$, rather than the uniaxial term $\frac{1}{2}\sigma_y^2/E$ used above, so that an allowance of $J/G_y = 3$ could be made in triaxial cases and this would be closely similar to that in the c.o.d. procedure. Dawes & Kamath (1978) argued that there was little scope for reduction below the present c.o.d. design curve in some residual stress cases but examination of their data suggests that the full allowance may be required only for certain very severe cases. The argument can obviously be extended in an *ad hoc* manner to combined mechanical and thermal stress problems. It must be reiterated that the treatment is heuristic and a more refined estimation of equivalent J from an area-modified integral has been proposed (Ainsworth *et al.* 1978), although whether the thermal state has the same characteristic J stress field as for mechanical stress is not clear. All the foregoing is based on evidence of onset of crack growth but still leaves open to question whether the value of J_c or δ_c used is the value 'at' onset of growth or a larger value after some slow growth has occurred.

Two treatments emerge. The first is the use of procedures intended to guard against initiation of crack growth while accepting the use of a value of toughness, be it J_c or δ_c or some other, that in fact allows for some growth. This procedure has been adopted by some proponents of c.o.d., and the recent better understanding of R-curve methods now permits some rationalization. The intention is to measure a value of apparent toughness after initiation of growth but before instability (e.g. at maximum load in a test that fails by ductile tearing), and to use that one value in a design curve or similar initiation procedure in the absence of any other points defining the R-curve. A fuller treatment is of course to measure a complete R-curve and estimate T_{app} for the structure, thereby conducting a complete instability analysis. The first step in either case is to ensure that the toughness data is either representative of the actual structural use or is a lower bound thereto, allowing for the effects of anisotropy, uncertainty of size of shear lip and of estimates of η, particularly for cracks growing in the through-thickness direction.

The case for using a maximum load toughness value has been discussed by Towers & Garwood (1979) in terms of crack stability. They argue that because of high constraint, the R-curve measured on a d.n.b. piece gives a lower bound, if due regard is paid to the effect of anisotropy and shear lip, and that because, on the restricted evidence of l.e.f.m., T_{app} for a load-controlled test (i.e. $\partial G/\partial a|_Q$) is lower for bending than for other configurations, then the maximum load toughness in terms of J or c.o.d. will be less than that found in practice. Expressed in the present terminology, using (13) if $w \gg f_1(\eta)$ (or more generally from (10) if that restriction is not made),

$$J_{eff} = (b/\eta)(\mathrm{d}J_r/\mathrm{d}a) \tag{25}$$

so that provided that

$$(b/\eta)_{component} > (b/\eta)_{test\text{-}piece}, \tag{26}$$

the same or greater value of J_{eff} will be reached before instability, provided also that the applied severity, measured in terms of either T_{app} or I, is less for the component than for the test-piece, i.e.

$$(T_{app})_{component} < (T_{app})_{test\text{-}piece}, \tag{27}$$

so that stability is maintained up to at least the same value of $\mathrm{d}J_r/\mathrm{d}a$. Full thickness data are also implied to avoid the possibility of fracture in the component under l.e.f.m. conditions of plane strain with no R-curve effect at all, despite yield in the test-piece. Equation (26) is seen to be satisfied provided that the ligament b for the component is greater than b for the test-piece because η for the d.n.b. (or more strictly the deep-notch c.t. piece which slightly exceeds the d.n.b. case) is larger than η for all the structural configurations yet examined, for both elastic and plastic behaviour.

A firm conclusion on (27) is less easy since, as already noted, estimates of T_{app} according to either the energetic argument (i.e. I) or the characterizing argument (i.e. $\partial J/\partial a$) differ in value. For the d.n.b. case $I = G$, since in (17) $\eta_0 \approx \eta_{el} \approx 2$, but for the c.c.t. case $I \approx G$ only for ideal e.p.e. (where $\eta_0 = \eta_{el}$ still), whereas for real material permitting a rigid plastic estimate, $\eta_0 \rightarrow 1$ and $I > G$ bounded only by $I < J$. Thus, subsequent to initiation, I for the component could increase for a tensile configuration as limit load is approached, and become larger than I for the test-piece (itself for the d.n.b. case), thereby promoting instability before the value J_{eff} found in the test-piece. In reality the effect of lower constraint on the R-curve and of the b/η term may increase J_{eff} for the tensile case and thus offset the higher value of I, but there seems no rigorous proof of this at the moment. The use of full-thickness pieces in either c.o.d. or J procedure while assessing the value of effective toughness beyond initiation will also guard against change of micro-mode in components of that thickness, whereas use of a small test-piece to determine effective toughness, even if justified on the above arguments, has not been demonstrated to ensure avoidance of a change in micro-mode in a thicker section. If a full analysis is made, the differences between the various approaches remain as a margin of uncertainty when $\omega \gg 1$ and for circumstances where there is doubt whether or not J is really meaningful. These points are not re-argued here. The concept of discussing stability of crack growth in structures is now available and the relevance of T_{app} and T_{mat} as the appropriate tools is clear, even if the numerical values to be used in a given case are not yet clear. For complex structural situations, Paris *et al.* (1979) has suggested an analysis treating a combination either in series or parallel, by which unstable tearing of any cracked component can be guarded against for overall changes in load or displacement applied to the whole structure. He remarks on the difficulty of assessing the possible magnitudes of load realistically and accepts that considering the verge of plastic collapse (for a pipe system subject, for example, to earthquakes) is adequate in that 'if a crack is stable under such conditions, then a plastic collapse problem, not a crack problem, controls'. The concept is called 'fracture-proof design' and is based on considering a statically indeterminate member that is cracked, where both it and some other members of the system are loaded to the fully plastic state just short of forming a collapse mechanism for the whole structure. A tearing analysis must then demonstrate the crack is still stable. Members in series are of course accommodated by treating the external compliance as an effective gauge length, in any of the instability analyses discussed. If strictly at fixed displacement, a parallel system does not affect the analysis for a single bar. Paris considers a fixed load applied to a system of n members in parallel, of which one is cracked, such that the cracked member and m others are at the limit state. He shows

$$T_{app} = [(n-m)/(n-1-m)]\,T_{nom}, \tag{28}$$

where T_{nom} is the conventional value of T_{app} of a single member under fixed displacement (as, for example, equation (16)). Restrictions such as the uncertainty over estimating T_{app}, small amounts of crack growth, avoidance of change in micro-mode, and neglect of time-dependent effects, all remain, of course, but again the concepts for a methodology of tearing instability analysis are now clear.

Conclusions

In discussing the severity of conditions applied to a sharp defect, the so-called J-contour integral is the most convenient and embracing parameter at present available. All other single-parameter descriptions of fracture on the continuum scale appear to be encompassed by it. Nevertheless, the J concept is not totally applicable to all circumstances since it loses strict relevance for incremental plasticity behaviour when there is a change in triaxiality or constraint from the datum case, which is universally agreed as l.e.f.m. conditions of plane strain, as the degree of plastic deformation increases. To account for this change in triaxiality, a second parameter would be required, the significance of which would depend upon the micro-mode of separation. It is not clear from the experimental evidence how far J or any other single term adequately describes the onset of fracture, and it is likely that the departure from strict applicability of J varies from one circumstance and material to another. Within some poorly defined limits there is, however, a practical usefulness of J or c.o.d. as a fracture parameter.

Recent understanding of ductile tearing suggests that the dominant term in ensuring stable growth is the rate of dissipation of plastic work, $\mathrm{d}w/B\mathrm{d}a$. This term can be normalized by means of size and shape factors b and η so that the material toughness is represented by $T_{\mathrm{mat}} = (E/\sigma_{\mathrm{y}}^2)(\mathrm{d}J/\mathrm{d}a)_{\mathrm{mat}}$. Onset of unstable ductile crack growth has been expressed as the exceeding of a balance between elastic deformation, q, or energy rates, I, or crack characterizing rate taken as $\mathrm{d}J/\mathrm{d}a$, and the material tearing toughness expressed in corresponding terms. All these concepts are formally identical for small amounts of crack growth described by $w \equiv (b/J)(\mathrm{d}J/\mathrm{d}a) \gg 1$, although the best numerical estimates differ according to the details of the theory adopted. The circumstances when ductile tearing may result in a change of micro-mode, for example to cleavage in steels, are still not clear.

For simple direct use, bypassing many of the above complexities, a J-based design curve has been derived. An approximate allowance for residual stress is included. The curve permits a simple estimate of the severity of the conditions applied to the crack tip in terms of the strain in the uncracked body analogous to the well known c.o.d. design curve. Selection of how severe are the worst conditions that must be resisted remains a problem-orientated question that cannot be generalized. Some rationalization in terms of the effective toughness, $(b/\eta)(\mathrm{d}J/\mathrm{d}a)$, is offered for the usage of an apparent toughness, beyond the true value at onset of growth, where a full instability analysis is not made. Analytical concepts to avoid tearing before some stated degree of plastic collapse occurs are, however, now emerging in what has been termed 'fracture-proof design', although the possibility of a change in micro-mode of separation is again not explicitly excluded. The circumstances when there can be a change in the micro-mode of behaviour was perhaps the original problem of brittle fracture in the engineering sense, and a better understanding of it in terms of continuum mechanics remains elusive.

References (Turner)

Ainsworth, R. A. *et al.* 1978 In *Tolerance of flaws in pressurised components*, pp. 197–204. London: Institution of Mechanical Engineers.
Begley, J. A. & Landes, J. D. 1972 In *Fracture toughness* (*A.S.T.M. S.T.P.* no. 514), pp. 1–23.
Begley, J. A., Landes, J. D. & Wilson, W. K. 1974 *Fracture Analysis* (*A.S.T.M. S.T.P.* no. 560), pp. 155–168.
Bilby, B., Cottrell, A. H., Swindon, K. H. & Smith, E. 1964 *Proc. R. Soc. Lond.* A **279**, 1–9.
Boyle, E. F. 1972 Ph.D. thesis. The Queen's University, Belfast.
Broberg, K. B. 1974 In *Prospects in fracture mechanics* (ed. G. Sih *et al.*), pp. 125–138. Noordhoff.
B.S.I. 1976 *Draft British Standards Rules for the derivation of acceptance levels for defects in fusion welded joints.* BSI WEE/37, Dec. 75/77081 D.C. London: British Standards Institution.
B.S.I. 1977 *Methods of test for plane strain fracture toughness* (K_{Ic}) *of metallic materials.* BS 5447. London: British Standards Institution.
B.S.I. 1979 *Methods for crack opening displacement (COD) testing.* BS 5762. London: British Standards Institution.
Burdekin, F. M. & Dawes, M. G. 1971 In *Practical applications of fracture mechanics to pressure vessel technology*, pp. 28–37. London: Institution of Mechanical Engineers.
Burdekin, F. M. & Harrison, J. W. 1979 In *Colloquium on practical application of fracture mechanics.* International Institute of Welding.
Cherepynov, G. P. 1967 *Appl. Math. Mech.* **31**, 503–512.
Dawes, M. G. 1974 *West. J. Res. Suppl.* **53**, 369s–379s.

Dawes, M. G. & Kamath, M. S. 1978 In *Conference on Tolerance of Flaws in Pressurized Components*, Inst. mech. Engrs paper no. C82/78.

Dugdale, D. S. 1960 *J. Mech. Phys. Solids* **8**, 100–104.

Egan, G. R. 1973 *Engng Fract. Mech.* **5**, 167–185.

Ernst, H., Paris, P. C., Parson, M. & Hutchinson, J. 1979 In *A.S.T.M. S.T.P.* no. 677, pp. 581–599.

Eshelby, J. D. 1956 In *Progress in solid state physics*, vol. 3, p. 79. New York: Academic Press.

Garwood, S. J. 1976 Ph.D. thesis, University of London.

Garwood, S. J. 1979 In *11th National Symposium on Fracture Mechanics* (*A.S.T.M. S.T.P.* no. 677), pp. 511–532.

Garwood, S. J. 1980 In *12th National Symposium on Fracture Mechanics* (*A.S.T.M. S.T.P.* no. 700). (In the press.)

Garwood, S. J., Pratt, P. L. & Turner, C. E. 1978 In *Elasto-plastic fracture mechanics* (*C.S.N.I. Report* no. 32; U.K.A.E.A.–O.E.C.D. (N.E.A.) vol. 1), paper 11.

Green, G. & Knott, J. 1975 *J. Mech. Phys. Solids* **23**, 167–185.

Griffis, C. A. 1975 Elastic–plastic fracture toughness: a comparison of J-integral and crack opening displacement characterisations. *J. Press. Vess. Technol.* **94**, 278–281.

Harrison, J. D. 1977 *Int. J. Press. Vess. Piping* **5**, 261–279.

Hayes, D. J. 1970 Ph.D. thesis, University of London.

Hutchinson, J. W. 1968 *J. Mech. Phys. Solids* **16**, 13–31.

Hutchinson, J. W. & Paris, P. C. 1977 *A.S.T.M. S.T.P.* no. 668, pp. 32–64.

Knott, J. 1973 *Fundamental of fracture mechanics*. London: Butterworth.

Krafft, J. M., Sullivan, A. M. & Boyle, R. W. 1961 Presented at Symp. on Crack Propagation, College of Aeronautics, Cranfield, Paper A1.

Landes, J. 1979 In *C.S.N.I. Report, O.E.C.D.–N.E.A. Specialists Meeting on Plastic Tearing Instability*. St Louis: Washington University.

Landes, J. & Begley, J. A. 1972 In *Fracture toughness*, (*A.S.T.M. S.T.P.* no. 514), pp. 24–39.

Latzko, D. G. H. (ed.) 1979 *Post-yield fracture mechanics*. London: Applied Science Publishers.

Logsdon, W. A. 1976 In *Mechanics of crack growth* (*A.S.T.M. S.T.P.* no. 590), pp. 43–60.

McClintock, F. 1965 *Proc. R. Soc. Lond.* A **285**, 58–74.

McClintock, F. 1968 In *Fracture: an advanced treatise* (ed. H. Liebowitz), vol. 3, ch. 2. New York: Academic Press.

McMeeking, R. M. & Parks, D. M. 1979 In *A.S.T.M. S.T.P.* no. 668), pp. 135–144.

Merkle, J. G. & Corten, H. T. 1974 *J. Press. Vess. Technol.* **96**, 200–229.

N.R.C. 1977 *A treatment of the subject of tearing instability* (ed. P. C. Paris, *et al.*). St Louis: Washington University NUREG 0311.

Orowan, E. 1956 *J. appl. Phys.* **26**, 900–902.

Paris, P. C., Gomez, M. P., Tada, H. & Baldini, S. E. 1979 In *12th National Symposium on Fracture Mechanics*. (In the press.)

Riccardella, P. C. & Swedlow, J. L. 1974 In *Fracture analysis* (*A.S.T.M. S.T.P.* no. 560), 134–154.

Rice, J. R. 1965 In *ICF*–1, Sendai, pp. A309–340.

Rice, J. R. 1968*a* *J. appl. Mech.* **35**, 379–386.

Rice, J. R. 1968*b* In *Fracture: an advanced treatise* (ed. H. Liebowitz), vol. 2, ch. 3. New York: Academic Press.

Rice, J. R. 1978 In *Numerical methods in fracture mechanics* (ed. A. Luxmoore & M. J. Owen), pp. 434–449. University College of Wales, Swansea.

Rice, J. R., Paris, P. C. & Merkle, J. G. 1973 In *Progress in flaw growth and toughness testing* (*A.S.T.M. S.T.P.* no. 536), pp. 231–245.

Rice, J. R. & Rosengren, G. F. 1968 *J. Mech. Phys. Solids* **16**, 1–12.

Robinson, J. N. 1976 *Int. J. Fract.* **12**, 723–729.

Robinson, J. N. & Tetelman, A. S. 1974 In *Fracture toughness and slow stable cracking* (*A.S.T.M. S.T.P.* no. 559), p. 139.

Shih, C. F. 1979 In *C.S.N.I. Report, O.E.C.D.–N.E.A. Specialists Meeting*. St Louis: Washington University.

Shih, C. F., de Lorenzi, H. G. & Andrews, W. R. 1979 In *Elastic plastic fracture* (*A.S.T.M. S.T.P.* no. 668), pp. 65–120.

Srawley, J. 1976 *Int. J. Fract.* **12**, 470–474.

Sumpter, J. D. G. 1973 Ph.D. thesis, University of London.

Sumpter, J. D. G. & Turner, C. E. 1976*a* *J. mech. Engng Sci.* **18**, 97–112.

Sumpter, J. D. G. & Turner, C. E. 1976*b* In *Cracks and fracture* (*A.S.T.M. S.T.P.* no. 601), pp. 3–18.

Sumpter, J. D. G. & Turner, C. E. 1976*c* *Int. J. Fract.* **12**, 861–871.

Towers, O. & Garwood, S. J. 1979 *Int. J. Fract.* **16**, R85–R90.

Turner, C. E. 1973 *Mat. Sci. Engng* **11**, 275–282.

Turner, C. E. 1978 In *Elasto–plastic fracture mechanics* (*C.S.N.I. report* no. 32, vol. 1. *U.K.A.E.A./O.E.C.D.*(*N.E.A.*). Paper.

Turner, C. E. 1979*a* In *A.S.T.M. S.T.P.* no. 677, pp. 614–628.

Turner, C. E. 1979*b* In *Colloquium on practical application of fracture mechanics*. International Institute of Welding.

Turner, C. E. 1979*c* In *Post-yield fracture mechanics* (ed. D. G. H. Latzko), ch. 2. London: Applied Science Publishers.

Turner, C. E. 1980*a* In *C.S.N.I. Report* no. 39 (*O.E.C.D.–N.E.A. Specialists' Meeting on Plastic Tearing Instability*). St Louis: Washington University.

Turner, C. E. 1980*b* In *Advances in elasto-plastic fracture mechanics* (ed. L. H. Larsson), pp. 301–318. London: Applied Science Publishers.

U.K.A.E.A. 1969 *Practical fracture mechanics for structural steel*. London: U.K.A.E.A./Chapman Hall.

Willoughby, A. A. 1979 Ph.D. thesis, University of London.

Willoughby, A. A., Pratt, P. L. & Turner, C. E. 1978 *Int. J. Fract.* **14**, R249–R251.

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